the new system find themselves able to do so without extra work. This applies to some public schools in New England and to a large number in other parts of the country. At present, there are only fourteen public high schools which have sent to Harvard College one boy a year for the past ten years, and all of these are in eastern Massachusetts.

These two results are primarily significant for the college. The other desired results, if they come about, are broader in their educational significance.

3. Schools of the approved type will, so far as Harvard College is concerned in the matter, gain the freedom which they require for doing their best work, since the new system will make it possible for them to concentrate their efforts by treating more thoroughly fewer subjects, or fewer topics of a subject. The great need of students in schools, as well as in colleges, is that they should acquire a habit of doing well what they undertake to do; the greatest evil in education at present is that students are satisfied with mediocrity.

4. The new system gives some help toward an adjustment of the problem of educating together in one school students preparing for college and students preparing for other callings. It does not wholly solve this problem, but it ought to tend somewhat to relieve it. The problem itself is insoluble. Preparation for a definite vocation must be determined by the character and needs of that particular vocation, and college is a vocation for a young man of seventeen to twenty-one just as much as service in a banking house or factory, and, like those vocations, it has its own conditions of fitness. Different needs can not all be provided for under one system of education. Nevertheless, some parts of a school course are an excellent preparation both for college and for an immediate practical career, and the new system of examinations, under which requirements in specific subjects are kept as high as before but the subjects less closely defined, will, it is hoped, give as much freedom here as the nature of the case permits.

5. The new plan leads away from emphasis on single courses, and insists on the significance of the education taken as a whole. In accord with this underlying idea it is free from all attempts to determine the relative value of subjects as expressed in numerical ratings. In this respect it has a general educational importance, and ought to remove many causes of friction now existing between schools and colleges.

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THE BOLYAI PRIZE. II

INTEGRAL EQUATIONS

In these latter years, Hilbert has above all occupied himself with perfecting the theory of integral equations. We know that the foundations of this theory were laid some years ago by Fredholm; since then the fecundity of his method and the facility with which it may be applied to all the problems of mathematical physics approve themselves each day with more luster. This is certainly one of the most remarkable discoveries ever made in mathematics, and for itself alone it would merit the very highest recompense; if to-day, however, it is not to the first inventor, but to the author of important improvements, that we have decided to award the Bolyai prize, it is because we must take into consideration not only Hilbert's works on integral equations, but the totality of his achievement, which is of importance for the most diverse branches of mathematical science and of which the other parts of this report permit us to appreciate the interest.

But we can not enter upon this subject without paying homage to the immense service which Fredholm has rendered to science.

The theory of Fredholm is a generalization of the elementary properties of linear equations and determinants. This generalization may be followed up in two different ways: on the one hand, by considering a *discrete* infinity of variables connected by an infinity of linear equations, which leads to determinants of infinite order; on the other hand, by considering an unknown function $\phi(x)$ (that is to say in last analysis a *continuous* infinity of unknowns) and seeking to determine it by the aid of equations where this function figures in integrals under the sign $\int$. Upon this second way Fredholm has embarked.

Let $K(x, y)$ be a function we call the *kernel;* the integral

$$\psi(x) = \int K(x, y)\phi(y)dy,$$

taken between fixed limits, may be regarded as a transform of $\phi(x)$ by a sort of linear transformation and be represented by $S\phi(x)$.

The integral equations may then be put under the form

$$(1) \qquad a\phi(x) + \lambda S\phi(x) = f(x),$$

where $f(x)$ is a given function; the equation is said to be of the first kind if the coefficient a is null, and of the second kind if this coefficient is equal to 1.

The relation (1) should be satisfied by all the values of y comprised in the field of integration; it is therefore equivalent to a *continuous* infinity of linear equations.

Fredholm has treated the case of the equations of the second kind; the solution then may be put under the form of the quotient of two expressions analogous to determinants and which are integral functions of λ. For certain values of λ, the denominator vanishes. We then can find functions $\phi(x)$ (called *proper functions*) which satisfy the equation (1) when we replace $f(x)$ in it by 0.

The result supposes that the kernel $K(x, y)$ is limited; if it is not, we are led to consider *reiterated* kernels; if we repeat n times the linear substitution S, we obtain a substitution of the same form with a different kernel $K_n(x, y)$; if one of these reiterated kernels be limited this suffices for the method to remain applicable by means of a very simple artifice. Now this happens in a great number of cases, as Fredholm has shown. The generalization for the case where the unknown function depends upon several variables and for that where there are several unknown functions is made without difficulty.

Fredholm then applied his method to the solution of Dirichlet's problem and to that of a problem in elasticity, thus showing how we may attack all questions of mathematical physics.

Such is the part of the first inventor; what now is Hilbert's? Consider first a finite number of linear equations; if the determinant of these equations is symmetric, their first members may be regarded as the derivatives of a quadratic form, and hence results for equations of this form a series of propositions very worthy of interest and well known to geometers. The corresponding case for integral equations is that where the kernel is symmetric, that is to say, where

$$K(x, y) = K(y, x).$$

This Hilbert takes hold of. The properties of quadratic forms of a finite number of variables may be generalized so as to apply to integral equations of this symmetric form. The generalization is made by a simple passing to the limit; but this passing presented difficulties which Hilbert overcame by a method admirable in its

simplicity, certainty and generality. The developments reached are *uniformly* convergent, but this uniformity presents itself under a new form which deserves to attract attention. In the developments appears an arbitrary function $u(x)$ (or several) and the remainder of the series when n terms have been taken is less than a limit depending only upon n and independent of the arbitrary function, provided this function is subject to the inequality

$$\int u^2(x)\,dx < 1,$$

the integral being taken between suitable limits. This is an entirely new consideration which may be utilized in very different problems.

Thus Hilbert obtains in a new way certain of Fredholm's theorems; but I shall stress above all the results which are most original.

In the first place, the denominator of Fredholm's expressions is a function of λ admitting only real zeroes, and this is a generalization of the elementary theorem relative to "the equation in S." Afterward comes a formula where enter under the sign $\int$ two arbitrary functions $x(s)$ and $y(s)$ which we should consider as the generalization of the elementary formulas which permit the breaking up of a quadratic form into a sum of squares.

But I hasten to reach the question of the development of an arbitrary function proceeding according to proper functions. Is this development, the analogue of Fourier's series or of so many other series playing a principal rôle in mathematical physics, possible in the general case? The sufficient condition that a function be capable of such development is that it can be put in the form $Sg(x)$, $g(x)$ being continuous. This is the final form of the resultant as Hilbert gives it in his fifth communication. In the first he was forced to impose certain restrictions; here we must mention the name of Schmidt, who in the interval had produced a work which helped Hilbert to free himself from these restrictions. The only condition imposed upon our function is capability of being put in the form $Sg(x)$, and at first blush this would seem sufficiently complex, but in a large number of cases and, for example, if the kernel is a Green's function, it only requires that the function possess a certain number of derivatives.

Hilbert was afterward led to develop his views in the following manner: he this time considers a quadratic form with an infinite number of variables and he studies its orthogonal transformations; this is as if he wished to study the different forms of the equation of a surface of the second degree in space of an infinite number of dimensions when referred to different systems of rectangular axes. To this effect he makes what he calls the resolvent form of the given form. Let $K(x)$ be the given form, $K(\lambda, x, y)$ the resolvent form sought; it will be defined by the identity

$$K(\lambda, x, y) - \tfrac{1}{2}\lambda\Sigma_r \frac{dK(x)}{dx_r}\,\frac{dK(\lambda, x, y)}{dx_r} = \Sigma x_r y_r.$$

When the form $K(x)$ depends only upon a finite number of variables, the resolvent form presents itself as the quotient of two determinants which are integral polynomials in λ.

Our author applies to this quotient the procedures of passing to the limit which are familiar to him; the limit of the quotient exists even when those of the numerator and of the denominator do not exist.

In the case of a finite number of variables, $K(\lambda, x, y)$ is a rational function of λ and this rational function can be broken up into simple fractions. What becomes of

this decomposition when the number of variables becomes infinite? The poles of the function rational in λ may in this case or otherwise tend toward certain limit points infinite in number but discrete.

The aggregate of these points constitutes what our author calls the *discontinuous spectrum* of his form. They may also admit as limit points all the points of one or several sects of the real axis. The aggregate of these sects constitutes the *continuous spectrum* of the form.

The simple fractions corresponding to the discontinuous spectrum will make in their totality a convergent series; those corresponding to the continuous spectrum will change at the limit into an integral of the form

$$\int \frac{\sigma d\mu}{\lambda - \mu},$$

where the variable of integration μ is varied all along the sects of the continuous spectrum, and where σ is a suitable function of μ. The rational function $K(\lambda, x, y)$, therefore, has then as limit not a meromorphic function, but a uniform function with erasures. The decomposition into simple elements thus transformed remains valid. If the given form is *limited,* that is to say, if it can not pass a certain value when the sum of the squares of the variables is less than 1, we can deduce thence a way of simplifying this form by an orthogonal transformation, analogous to the simplification of the equation of an ellipsoid by referring this surface to its axes.

Among the quadratic forms we shall distinguish those which are *properly continuous* (vollstetig), that is to say, those whose increment tends toward zero when the increments of the variables tend simultaneously toward zero in any way. Such a form does not have a continuous spectrum and hence result noteworthy simplifications in the formulas.

Other theorems on the systems of simultaneous quadratic forms, on bilinear forms, on Hermite's form, extend likewise to the case of an infinite number of variables.

There was in this theory the germ of an extension of Fredholm's method to kernels to which the analysis of the Swedish geometer was not applicable, and scholars of Hilbert should bring out this fact. However that may be, Hilbert first applied himself to extending his way of looking at integral equations to the cases where the kernel is unsymmetric. For this purpose he introduces any system of orthogonal functions, conformably to which it is possible to develop an arbitrary function by formulas analogous to that of Fourier. In place of an unknown function, he takes as unknowns the coefficients of the development of this function; an integral equation can thus be replaced by a system of a *discrete* infinity of linear equations between a *discrete* infinity of variables.

The theory of integral equations is thus attached, on the one hand, to the ideas of von Koch on infinite determinants, and, on the other hand, to the researches of Hilbert we have just analyzed and where the essential rôle is played by functions dependent upon a discrete infinity of variables.

To each kernel will correspond thus a bilinear form dependent upon an infinity of variables. If the kernel is symmetric, this bilinear form is symmetric and may be regarded as derived from a quadratic form. If the kernel satisfies the conditions stated by Fredholm, we see that this quadratic form is properly continuous and consequently does not have a continuous spectrum. This is a way of reaching Fredholm's results, and however indirect it may be, it opens entirely new views of the profound reasons for these results and hence on the possibility of new generalizations.

Integral equations lend themselves to the

solution of certain differential equations whose integrals are subject to certain conditions as to the limits, and this is a very important problem for mathematical physics. Fredholm solved it in certain particular cases and Picard generalized his methods. Hilbert made a systematic study of the question.

Consider an integral equation

$$\Delta u = f,$$

where u is an unknown function of one or several variables, f a known function and Δ any linear differential expression. This equation with the same right as an integral equation may be considered as an infinite system of linear equations connecting a continuous infinity of variables, as a sort of linear transformation of infinite order, enabling us to pass from f to u. If we solve this equation, we find

$$u = Sf,$$

$S(f)$ this time presenting itself under the form of an integral expression.

Then Δ and S are the symbols of two linear transformations of infinite order inverse one to the other. The kernel of this integral expression $S(f)$ is what we call a *Green's function.* This function was first met in Dirichlet's problem, then it was Green's function properly so called, too familiar to be stressed; we had already obtained different generalizations of it. To have given a complete theory belongs to Hilbert. To each differential expression Δ, supposed of the second order and of elliptic type, to each system of conditions as to the limits, corresponds a Green's function. We cite the formation of the Green's functions in the case where we have only one independent variable and where they present themselves under a particularly simple form, and the discussion of the different forms the conditions as to the limits may assume. That settled, suppose we have solved the problem in the case of an auxiliary differential equation differing little from that proposed and anyhow not differing from it by the terms of the second order; we can then by a simple transformation reduce the problem to the solution of a Fredholm equation where the rôle of kernel is played by a Green's function relative to the auxiliary differential equation. However, the consideration of this auxiliary equation, the necessity of choosing it and solving it being capable of still constituting an embarrassment, in his sixth communication Hilbert frees himself from it. The differential equation is transformed into a Fredholm equation where the rôle of kernel is played by a function our author calls *parametrix.* It is subject to all the conditions defining Green's function, one alone excepted, the most troublesome, it is true; it is not constrained to satisfy a differential equation; it remains therefore in a very large measure arbitrary. The transformation undergone by the differential equation is comparable to that experienced by a system of linear equations if we replace the primitive variables by linear combinations of these variables suitably chosen. The method is nowise restricted to the case where the differential equation considered is adjoint to itself.

Hilbert examined in passing a host of questions relative to integral equations and showed the possibility of their application in domains the most varied. For example, he extended the method to the case of a system of two equations of partial derivatives of the first order of the elliptic type, to *polar* integral equations, that is to say, where the coefficient a in the integral equation (1) in place of being always equal to 1 is a function of x and in particular is equal now to $+1$, now to -1.

He has applied the method to the problem of Riemann for the formation of func-

tions of a complex variable subject to certain conditions as to the limits, to the theorem of oscillations of Klein, to the formation of fuchsian functions, and in particular to the following problem: to determine λ so that the equation

$$\frac{d}{dx}\left[(x-a)(x-b)(x-c)\frac{dy}{dx}\right]+(x+\lambda)y=0$$

may be a fuchsian equation.

One of the most unexpected applications is that Hilbert makes to the theory of the volumes and surfaces of Minkowski, and by which he connects with Fredholm's method a question important for those who interest themselves in the philosophic analysis of the fundamental notions of geometry.

DIRICHLET'S PRINCIPLE

We know that Riemann with a stroke of the pen proved the fundamental theorems of Dirichlet's problem and conformal representation, grounding himself on what he called Dirichlet's principle; considering a certain integral depending upon an arbitrary function U, and which we shall call Dirichlet's integral, he showed that this integral can not become null and from this he concludes that it must have a minimum, and that this minimum can be reached only when the function U is harmonic. This reasoning was faulty, as has since been shown, because it is not certain that the minimum can be actually reached, and if it is, that it can be for a continuous function.

Yet the results were exact; much work has been done on this question; it has been shown that Dirichlet's problem can always be solved, and it actually has been solved; it is the same with a great number of other problems of mathematical physics which formerly would have seemed attackable by Riemann's method. Here is not the place to give the long history of these researches; I shall confine myself to mentioning the final point of outcome, which is Fredholm's method.

It seemed that this success had forever cast into oblivion Riemann's sketch and Dirichlet's principle itself. Yet many regretted this; they knew that thus we were deprived of a powerful instrument and they could not believe that the persuasive force which in spite of all Riemann's argument retained, and which seemed to rest upon I know not what adaptation of mathematical thought to physical reality, was actually only a pure illusion due to bad habits of mind. Hilbert wished to try whether it would not be possible, with the new resources of mathematical analysis, to turn Riemann's sketch into a rigorous proof. See how he arrived at it; consider the aggregate of functions U satisfying proposed conditions; choose in this aggregate an indefinite series of functions S, such that the corresponding Dirichlet integrals tend in decreasing toward their lower limit. It is not certain that at each point of the domain considered this series S is convergent; it might oscillate between certain limits. But we can in S detach a partial series S_1 which is convergent at a point M_1 of the domain; in S_1, detach another partial series S_2 which shall always be convergent at M_1, but which, moreover, shall also be convergent at M_2. So continuing, we shall obtain a series which will be convergent at as many points as we wish; and by a simple artifice we from this deduce another series which will be convergent at all the points of a countable assemblage, for example at all the points whose coordinates are rational. If then we could prove that the derivatives of all the functions of the series are less in absolute value than a given limit, we might conclude immediately that the series converges uniformly in the whole domain and the application of the rules of the calculus

of variations would no longer present special difficulty.

To establish the point remaining to be proved, Hilbert has used two different artifices; he has not developed the first as completely as would be desirable, and has attached himself especially to the second. This consists in replacing the proposed function u by the function v, which comes from it by a double quadrature and of which it is the second derivative with regard to two independent variables. The derivatives of v being the first integrals of u, we can assign them an upper limit, by the help of certain inequalities easy to prove. Only it is necessary to be resigned to a new circuit and to an artifice simple however to apply to this new unknown function v the rules of the calculus of variations which apply so naturally to the function u.

It is needless to insist upon the range of these discoveries which go so far beyond the special problem of Dirichlet. It is not surprising that numerous investigators have entered the way opened by Hilbert. We must cite Levi, Zaremba and Fubini; but I think we should signalize before all Ritz, who, breaking away a little from the common route, has created a method of numeric calculus applicable to all the problems of mathematical physics, but who in it has utilized many of the ingenious procedures created by his master Hilbert.

Recently Hilbert has applied his method to the question of conformal representation. I shall not analyze this memoir in detail. I shall confine myself to saying that it supplies the means of making this representation for a domain limited by an infinite number of curves or for a simply connected Riemann surface of an infinity of sheets. This therefore is a new solution of the problem of the uniformization of analytic functions.

DIVERS

We have passed in review the principal research subjects where Hilbert has left his trace, those for which he shows a sort of predilection and whither he has repeatedly returned; we must mention still other problems with which he has occupied himself occasionally and without insistence. I think I should confine myself to giving in chronologic order the most striking results he has obtained of this sort.

Excepting the binary forms, the quadratic forms and the biquadratic ternary forms, the definite form most general of its degree can not be broken up into a sum of a finite number of squares of other forms.

By elementary procedures may be found the solutions in integers of a diophantine equation of genus null.

If an integral polynomial depending upon several variables and several parameters is irreducible when these parameters remain arbitrary, we may always give these parameters integral values such that the polynomial remains irreducible.

Consequently there always exist equations of order n with integral coefficients and admitting a given group.

The fundamental theorem of Dedekind about complex numbers with commutative multiplication may be easily proved by means of one of the fundamental lemmas of Hilbert's theory of invariants.

The diophantine equation obtained by equating to ± 1 the discriminant of an algebraic equation of degree n has always rational solutions, but save for the second and the third degrees has no integral solutions.

Among the real surfaces of the fourth order, certain forms logically conceivable are not possible; for example, there can not be any composed of twelve closed surfaces simply connected or of a single surface with eleven perforations.

CONCLUSIONS

After this recital, a long commentary would be useless. We see how great has been the variety of Hilbert's researches, the importance of the problems he has attacked. We shall signalize the elegance and the simplicity of the methods, the clearness of the exposition, the solicitude for absolute rigor. In seeking to be perfectly rigorous one risks at times being long, and this is not to buy too dear a correctness without which mathematics would be nothing. But Hilbert has known how to avoid the tedium of such diffuseness for his readers in never letting them lose from view the guiding thread which has served him to orient himself. We always easily see by what chain of ideas he has been led to set himself a problem and find its solution.

We realize that, more analyst than geometer in the ordinary sense of the word, he nevertheless has seen at one view the totality of his work before distinguishing details and he knows how to give his reader the advantage of this all-embracing vision.

Hilbert has had a tremendous influence upon the recent progress of the mathematical sciences, not alone by his personal work, but by his teaching, by the counsel he has given to his scholars and which has enabled them to contribute in their turn to this development of our knowledge by using the methods created by their master.

There is no need, so it seems, to say more in justification of the decision of the commission which has unanimously awarded to Hilbert the Bolyai prize for the period 1905–1909.

M. POINCARÉ

THE WILLARD GIBBS MEDAL

IN the early part of 1909 Mr. William Converse, of Chicago, proposed to the Chicago Section of the American Chemical Society to found a gold medal to be awarded annually by the Section. Mr. Converse stated that the object of his proposition was to stimulate interest in the work of the Section and of the society at large and to encourage the highest ideals of the science in their members. The Section gladly welcomed and accepted the offer made. It was proposed to name the medal after the most eminent chemist America has given to the science, and the consent of Mrs. Van Name, the surviving sister of Willard Gibbs, having been secured, the medal founded by Mr. Converse was named the Willard Gibbs Medal. After various plans had been suggested and discussed, the Section decided that the medal should be awarded annually, by invitation, rather than by competition and the following rules were adopted for the award.

RULES FOR THE AWARD OF THE WILLARD GIBBS MEDAL, FOUNDED BY WILLIAM A. CONVERSE

1. A gold medal shall be awarded annually by the Chicago Section of the American Chemical Society at its May meeting, which meeting shall be open to the public.

The medal is to be known as the Willard Gibbs Medal founded by William A. Converse.

The award shall be made according to the rules here set forth and made a part of the by-laws of the Chicago Section.

2. The award shall be made by a two-thirds vote of a jury of twelve, to anybody who because of his eminent work in and original contributions to pure or applied chemistry, is deemed worthy of special recognition by the jury.

3. A condition of the award shall be that the recipient of the medal shall deliver an address upon a chemical subject of his own selection and satisfactory to the jury at the May meeting of the Chicago Section of the American Chemical Society. He shall be notified of the award three months in advance of this meeting by the chairman of the Chicago Section.

4. The jury of the award, to be known as the Jury of the Willard Gibbs Medal, shall consist of twelve members, six of them to be members of the Chicago Section. The chairman of the Chicago Section shall be chairman of the jury, but shall have no vote.

5. Four members of the jury shall be elected each year to serve three years, in the same manner